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Statistical energy analysis: coupling loss factors, indirect coupling and system modes

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Abstract

The distribution of energy among the subsystems of a system can be found in terms of the modes of the system. If there are enough modes in the frequency band of interest then the system can be described by an SEA model. However, in general this is a "quasi-SEA" model, which involves both direct and indirect coupling loss factors, whose values depend on the modal overlap. This paper explores the conditions under which the indirect coupling loss factors are zero, so that the system is described by a "proper-SEA" model. It also investigates the dependence of the direct and indirect coupling loss factors on the modal properties of the system and on the modal overlap. In summary, the indirect coupling loss factors are zero, so that a proper-SEA model can be formed, either if all the system modes are local or in the weak coupling regime as the modal overlap becomes large. It is seen that in the low modal overlap limit the coupling loss factors are proportional to the damping loss factor and equipartition of energy only occurs if all modes are global. For higher modal overlap the SEA parameters depend on the detailed statistics of the modes and the situation is complicated. However, if all the modes are local the indirect coupling loss factors are all zero. In the high modal overlap limit the coupling loss factors asymptote to constants, with indirect coupling loss factors becoming zero. This behaviour occurs because of mode shape correlation at the boundary between two directly connected subsystems.

1. Introduction

Energy-based modelling approaches are often used to describe the higher frequency vibrational behaviour of complex systems in some average, statistical or approximate way. The most important of these methods is statistical energy analysis (SEA) [1,2]. The system is divided into subsystems which are excited by random, stationary, distributed forces. The response is described

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by the time-average input powers $\mathbf{P}_{in}$ and the subsystem energies $\mathbf{E}$ averaged over some frequency band Ω. The coupling power between two subsystems is assumed to be proportional to the difference in their modal energies, the constants of proportionality being related to the coupling loss factors. This is the so-called assumption of coupling power proportionality (CPP). However, SEA involves a number of assumptions and approximations, whose validity and accuracy are usually unknown, and these restrict the applicability of SEA and the confidence that the analyst can place in its predictions.

The original theoretical approaches to SEA were along modal lines. CPP is known to be exact for two coupled oscillators under broadband excitation [3–5], for two coupled identical sets of oscillators [5], and for weakly coupled, lightly damped subsystems of oscillators under broadband excitation [6]. Extending the result to the general case of coupled multi-modal subsystems requires various assumptions to be made [1–4]. An ensemble of systems is introduced, with coupling power proportionality relating the ensemble average coupling powers and subsystem modal energies. The ensemble was defined to be such that the subsystem natural frequencies were independent random variables with a uniform probability distribution. The most important assumptions required are that (i) the interaction between two modes in two different subsystems is unaffected by the presence of additional modes in those subsystems and (ii) the interaction between two subsystems is unaffected by the presence of a third subsystem. Langley [7,8] showed that under certain conditions the dynamics of a certain broad class of system can be cast in SEA form and that under weak coupling conditions these equations reduce to the standard SEA equations. Finally, in Ref. [9] the modes of the system as a whole were used to determine the subsystem energies. It was seen that an SEA-like model of the system can always be formed if there are "enough" modes in the frequency band of interest, but that there may be non-zero indirect coupling loss factors.

The most common method for estimating coupling loss factors involves a wave approach [1,2,4,5,8]. Diffuse, incoherent incident wavefields are assumed, the subsystems assumed to be reverberant and the effects of reflections in the subsystems ignored. Again, these assumptions (particularly that of incoherence) are of unknown validity. However, these assumptions have been removed in Refs. [10,11], in which CPP has been shown to hold for the cases of two, uniform, coupled one-dimensional subsystems [10] and for two, coupled, simply supported rectangular plates [11] when appropriately ensemble averaged. Furthermore, the coupling loss factors were seen to depend on the modal overlap.

While SEA has grown to become a valuable engineering tool, there remain concerns regarding its applicability and accuracy. In this paper many of the assumptions and approximations made in conventional SEA theory are removed or relaxed. A modal approach is adopted, using results from Ref. [9], and the conditions under which a proper, conventional SEA model can be formed are investigated. Two questions are investigated. If an SEA model is formed, must indirect coupling loss factors be included? What values should be used for the coupling loss factors, and how do they depend on the modal overlap M? In summary, the answers are as follows. Indirect coupling loss factors are negligible in two circumstances: if all the system modes are local or in the limit of high modal overlap. The direct and indirect coupling loss factors are proportional to the damping loss factor at low M, but asymptote to constants at high M.

The concept of indirect coupling loss factors has been introduced elsewhere [1,7–9,12,13]. Numerical examples were considered in Refs. [12,13], while Refs. [7,8] contain general discussions.

The fact that at low modal overlap the coupling loss factors become proportional to the damping loss factor has been observed elsewhere [10–12,14,15].

The remainder of this section concerns various background issues, while relevant results from Ref. [9] are reviewed in Section 2. Sections 3–5 consider the existence of proper-SEA models and the values of the SEA parameters for low, increasing and high modal overlap. Section 6 concerns the case where the system comprises just 2 subsystems, while Section 7 contains some examples.

1.1. Terminology: ED and SEA models, modes and strength of coupling

In an energy distribution (ED) model the subsystem energies and input powers are related by

$$\mathbf{E} = \mathbf{A}\mathbf{P}_{in}, \tag{1}$$

where $\mathbf{A}$ is a matrix of energy influence coefficients (a list of symbols is given in Appendix B). In this paper an ED model is formed in terms of the modes of the system with few assumptions being made. In an SEA model, on the other hand,

$$\mathbf{P}_{in} = \mathbf{L}\mathbf{E}, \tag{2}$$

where the elements of $\mathbf{L}$ have the special form

$$L_{jj} = \omega\left(\eta_j + \sum_{i=1,N_s} \eta_{ji}\right), \quad L_{ij} = -\omega\eta_{ji}. \tag{3}$$

Here η_j and η_{ji} are damping and coupling loss factors and N_s is the number of subsystems. The matrix $\mathbf{L}$ must satisfy two conditions in order to be an SEA matrix. The first is *conservation of energy*: the jth column of $\mathbf{L}$ must sum to $\omega\eta_j$. The second is that the elements of $\mathbf{L}$ must satisfy the *consistency relation*

$$n_i\eta_{ij} = n_j\eta_{ji}, \tag{4}$$

where n_i is the asymptotic modal density of subsystem i. The coupling loss factors of an SEA model might also satisfy certain *desirable* conditions. In particular, they should be zero if the two subsystems are not physically connected, i.e., all indirect coupling loss factors should be zero. It is also usual in conventional SEA that they should be positive, independent of damping and depend only on the local system properties.

SEA involves various assumptions and approximations, and a particular system may not be able to be modelled accurately using SEA so that the inverse $\mathbf{X} = \mathbf{A}^{-1}$ need not have the structure of an SEA-matrix $\mathbf{L}$. While $\mathbf{X}$ always satisfies conservation of energy, only under some circumstances does it satisfy the consistency relation ([9], and Section 2.2).

If the necessary conditions (conservation of energy and consistency) are satisfied and also the indirect coupling loss factors are zero, $\mathbf{X}$ is said here to be a "proper-SEA" matrix. The coupling power between two subsystems is then proportional to the difference of their modal energies. If only the necessary properties are satisfied, $\mathbf{X}$ is said to be a "quasi-SEA" model: the system may still be modelled using an SEA-like approach, but indirect coupling loss factors are required to accurately model the system's response. This paper concerns the existence of proper-SEA models and the parameters of proper-SEA and quasi-SEA models.

It is worth emphasizing here that the existence of a non-zero indirect coupling loss factor η_{ri} does *not* imply that energy flows between subsystems r and i (which are not physically coupled). Instead, it means that the coupling power P_{rs} between two subsystems r and s which are physically coupled depends also on the energy of the third subsystem i. Hence the SEA assumption of coupling power proportionality does not hold. The existence of indirect coupling loss factors thus indicates that some of the assumptions of conventional SEA breakdown.

Finally, the remaining desirable conditions may not be satisfied (e.g., the coupling loss factors may depend on the damping or on global properties). This is more an issue regarding whether the coupling is strong or weak.

In this paper the term *system mode* is used to describe a mode of the assembled system, while a *subsystem mode* is one of a subsystem when uncoupled from the remainder of the structure. A *global mode* is a system mode which is global in the sense that the kinetic energy of the mode is spread out globally through the system. A *local mode*, however, is a system mode which is localized within a region of the system, so that the kinetic energy tends to be contained primarily in one subsystem and partly in its immediate neighbours (see Section 3.1.2).

The terms *weak coupling* and *strong coupling* are used in an SEA sense, although there is no universal, commonly accepted definition of what this means as yet. In contrast, two subsystems can be described as being *strongly* (or *weakly*) *connected* if energy can (or cannot) flow freely across the interface between them. Typically, the interface has a large transmission coefficient or there is a small impedance mismatch across it. Generally, weak and strong connections are associated with there being predominantly local or global modes, respectively. A clear distinction is thus made in this paper between the SEA strength of coupling and the strength of connection, a distinction which is not normally made in the usual wave approach to SEA. That there is a distinction will be seen in Section 3.

The modal overlap of the system as a whole is

$$M = n_{tot}\Delta \approx \frac{N\Delta}{\Omega}, \tag{5}$$

where n_{tot} is the total (asymptotic) modal density of the system, Δ is the (average) modal bandwidth and N is the number of modes in the band of excitation Ω. The bandwidth $\Delta = \omega\eta$, where ω and η are frequency and damping loss factors, respectively. Thus as the level of damping increases so too does the modal overlap.

2. Modes, energy and SEA

This section contains a brief summary of how energy influence coefficients, and from them coupling loss factors, can be written in terms of the modes of the system. Further details are given in Ref. [9] and in Appendix A. The remainder of the paper concerns how these parameters depend on the modal overlap M.

There are two necessary assumptions, namely that the system is linear and the excitations applied to the different subsystems are uncorrelated. Two further assumptions are then made: the excitation is “rain-on-the-roof”, defined to be random, spatially delta-correlated and with a spectral density that is independent of frequency, but varies spatially in proportion to the local mass density; the damping is light and proportional. The following assumptions are also made, primarily for convenience: all modes have equal bandwidth $\Delta = \omega\eta$; the time-average kinetic and potential energies are equal; the response is dominated by resonant modes. It is also assumed that the bandwidth of excitation, Ω, is large enough to contain "enough" modes of the system, as discussed in Section 2.2 below.

Suppose the jth mode has a natural frequency ω_j and a mode shape $\phi_j(\mathbf{x})$. Here, $\mathbf{x}$ and ϕ_j may be one-, two- or three-dimensional vectors depending on the structure. The energy influence coefficient A_{rs}, which gives the total energy in subsystem r per unit power input to subsystem s, is given by

$$A_{rs} = \frac{E_r}{P_{in,s}} = \frac{1}{\omega\eta} \frac{\sum_j \sum_k \Gamma_{jk} \psi_{jk}^{(s)} \psi_{jk}^{(r)}}{\sum_j \Gamma_{jj} \psi_{jj}^{(s)}}, \tag{6}$$

where the frequency integral

$$\Gamma_{jk} = \frac{1}{\Omega} \int_{\omega \in \Omega} \tfrac{1}{4} \omega^2 \, \mathrm{Re}\{\alpha_j(\omega) \alpha_k^*(\omega)\} \, \mathrm{d}\omega \tag{7}$$

depends on the natural frequencies and bandwidths of the modes in the excitation band Ω. Since it depends on the level of damping it therefore depends on the modal overlap. Here

$$\alpha_j(\omega) = \frac{1}{(\omega_j^2(1 + \mathrm{i}\eta) - \omega^2)} \tag{8}$$

is the modal receptance of the jth mode.

The cross-mode participation factor

$$\psi_{jk}^{(r)} = \int_{x \in r} \rho(\mathbf{x}) \phi_j(\mathbf{x}) \phi_k(\mathbf{x}) \, \mathrm{d}\mathbf{x} \tag{9}$$

depends on the mode shapes of the jth and kth modes within subsystem r, but is independent of the modal overlap.

2.1. Discussion and interpretation

2.1.1. Frequency integrals Γ_{jk}

The frequency integral Γ_{jk} is a term whose magnitude depends primarily on the natural frequencies of modes j and k and how close are these natural frequencies. An analytical expression for the integral is given in Appendix A. The "self-term" Γ_{jj} is always large if mode j is resonant, i.e., if ω_j lies in the band of excitation Ω. The "cross-term" Γ_{jk} ($j \neq k$) is also large if modes j and k are both resonant and if they *overlap*, i.e., they lie within each others half-power bandwidths such that $|\omega_j - \omega_k| \leqslant \Delta$.

If the damping is small then, for resonant modes, the frequency limits of integration in Eq. (7) can be extended to (0, ∞) to a good approximation. For non-resonant modes Γ_{jk} is negligible and such modes can therefore be ignored. The sums in Eq. (6) are then taken only over those modes which lie in Ω. For resonant modes then as a good approximation

$$\Gamma_{jj} = \Gamma_{kk} = \frac{1}{\Omega}\frac{\pi}{8\Delta},$$
$$\Gamma_{jk} = \Gamma_{jj}\mu_{jk},$$
$$\mu_{jk} = \frac{M_{jk}^2}{1+M_{jk}^2}, \quad M_{jk} = \frac{\Delta}{|\omega_j - \omega_k|}. \tag{10}$$

Here M_{jk} is the modal overlap of the two modes (i.e., the ratio of the bandwidth to the modal spacing). The term μ_{jk} acts as a filter that determines which mode pairs contribute significantly to the response: μ_{jk} is close to unity if the modes overlap, but is negligible if they do not overlap. As the modal overlap increases (and hence the loss factor and bandwidth Δ increase), the value of Γ_{jk} for an increasing number of mode pairs becomes significant and hence more mode pairs contribute to the sum in the numerator of Eq. (6). It is this, together with the statistics of $\psi_{jk}^{(r)}$, which causes the parameters of ED and SEA models to vary with modal overlap.

2.1.2. *Cross-mode participation factors*

The cross-mode participation factor $\psi_{jk}^{(r)}$ indicates the correlation of the $(j-k)$th mode pair within subsystem r. The self-term $\psi_{jj}^{(r)}$ gives the proportion of the kinetic energy of mode j which is stored in subsystem r and indicates the degree to which the jth mode is localized within that subsystem. Global modes of the system are those for which $\psi_{jj}^{(r)}$ is substantial for all (or at least many) of the subsystems, while local modes are those for which $\psi_{jj}^{(r)}$ is non-negligible in only one (or a few) of the subsystems. Every mode is orthogonal over the whole system, and therefore if there are N_s subsystems

$$\sum_{s=1}^{N_s} \psi_{jk}^{(s)} = \delta_{jk}. \tag{11}$$

The self-terms $\psi_{jj}^{(r)}$ are necessarily positive, while the cross-terms may be negative.

Large contributions to the input power come from modes j for which the participation factor $\psi_{jj}^{(s)}$ is large [9], that is, from those modes which respond well (i.e., for which the mode shape is large) in the excited subsystem. Mode pairs for which both $\psi_{jk}^{(s)}$ and $\psi_{jk}^{(r)}$ are large tend to give large contributions to the energy in subsystem r. These are modes that are both well-excited and which respond well. The input power depends on how many modes are excited, while the kinetic energy depends also on how well these modes overlap.

2.2. *The existence of SEA-like models*

In Ref. [9] it was seen that, if all modes have equal bandwidth and the contribution of out-of-band modes is small, the inverse $\mathbf{X} = \mathbf{A}^{-1}$ is an SEA-like matrix if the condition

$$\overline{\psi_{jj}^{(r)}} = \mathrm{E}[\psi_{jj}^{(s)}] \tag{12}$$

holds, where $\overline{\psi_{jj}^{(r)}}$ is the frequency average over all the modes in the frequency band Ω and $\mathrm{E}[\cdot]$ denotes the expectation, or the asymptotic average, over many modes. This expectation is given by

$$\mathrm{E}[\psi_{jj}^{(s)}] = \nu_s = \frac{n_s}{n_{tot}}, \tag{13}$$

where ν_s is the fractional modal density of subsystem s, i.e., the ratio of the asymptotic modal density n_s of subsystem s to the total modal density of the system.

If the above condition holds then an SEA-like model of the system can be formed: it may be a proper-SEA model, or it may be a quasi-SEA model and involve indirect coupling loss factors. Furthermore, if Eq. (12) holds, it does so *irrespective of the level of damping*, so that if an SEA-like model exists, it exists whatever the level of damping or modal overlap, or whether the coupling is strong or weak, and the parameters (direct and indirect coupling loss factors) depend on the modal overlap. This dependency, together with the existence of a proper-SEA model, will be explored in the rest of this paper.

Henceforth, it will be assumed that the band Ω is wide enough such that the modes within it satisfy Eq. (12). The sums over j in Eq. (6) can then be replaced by their expectations, so that the energy influence coefficients become

$$A_{rs} = \frac{1}{\omega\eta} \frac{\mathrm{E}\left[\sum_k \mu_{jk}\psi_{jk}^{(r)}\psi_{jk}^{(s)}\right]}{\nu_s}. \tag{14}$$

In effect, the sum over k runs over all the modes in Ω since it is assumed to be dominated by the contributions from the resonant modes, and in particular from those mode pairs which overlap (for which μ_{jk} is significant).

Various other expressions will be of use. The inverse of **A** can be written as

$$\mathbf{A}^{-1} = \omega\eta\mathbf{I} + \omega\mathbf{C}, \tag{15}$$

where **C** is a matrix of coupling loss factors with

$$\begin{aligned} C_{rs} &= -\eta_{sr}, \quad s \neq r, \\ C_{rr} &= \sum_{s,s\neq r} \eta_{sr}. \end{aligned} \tag{16}$$

A can also be written as

$$\mathbf{A} = \frac{1}{\omega\eta}(\mathbf{I} - \boldsymbol{\alpha}), \tag{17}$$

where

$$\alpha_{rs} = \delta_{rs} - \frac{\mathrm{E}\left[\sum_k \mu_{jk}\psi_{jk}^{(r)}\psi_{jk}^{(s)}\right]}{\nu_s} \tag{18}$$

while **C** can be written as

$$\mathbf{C} = \eta(\boldsymbol{\alpha} + \boldsymbol{\alpha}^2 + \boldsymbol{\alpha}^3 + \cdots). \tag{19}$$

If terms of higher order than the first can be neglected then

$$\eta_{sr} = -\eta\alpha_{rs} = \frac{\mathrm{E}\left[\sum_k \mu_{jk}\psi_{jk}^{(r)}\psi_{jk}^{(s)}\right]}{v_r}. \tag{20}$$

3. Low modal overlap

As the modal overlap tends to zero, $\Gamma_{jk} \to \Gamma_{jj}\delta_{jk}$ and cross-modal interactions become negligible. The coupling becomes strong in the SEA sense. The energy influence coefficients and elements of the matrix $\boldsymbol{\alpha}$ become

$$A_{rs} = \frac{1}{\omega\eta}\frac{\mathrm{E}\left[\psi_{jj}^{(r)}\psi_{jj}^{(s)}\right]}{v_s},$$
$$\alpha_{rs} = -\frac{\mathrm{E}[\psi_{jj}^{(r)}\psi_{jj}^{(s)}]}{v_s}, \quad r \neq s,$$
$$\alpha_{ss} = \frac{\mathrm{E}\left[\sum_{r\neq s}\psi_{jj}^{(r)}\psi_{jj}^{(s)}\right]}{v_s}, \tag{21}$$

where the last result is found using Eq. (11). Only the 'self' modal terms $\psi_{jj}^{(r)}$ are involved. A mode for which $\psi_{jj}^{(r)}\psi_{jj}^{(s)}$ is large gives a large contribution to A_{rs}. Such a mode is both strongly excited in subsystem s and responds well in subsystem r. The response parameters are determined solely by the statistics of the self-term $\psi_{jj}^{(r)}$, whose mean value is v_r. The expectations in Eq. (21) are

$$\mathrm{E}[\psi_{jj}^{(r)2}] = v_r^2 + \sigma_r^2,$$
$$\mathrm{E}[\psi_{jj}^{(r)}\psi_{jj}^{(s)}] = v_r v_s + \rho_{rs}, \tag{22}$$

where σ_r^2 and ρ_{rs} are the variance and covariance of the modal participation factors. The specific values of these quantities depend on the detailed statistics of the system mode shapes.

3.1. Coupling loss factors

Since the expectations in Eq. (21) are independent of η, $\mathbf{A}$ is inversely proportional to η, while $\boldsymbol{\alpha}$ is independent of η. Thus both $\mathbf{A}^{-1}$ and the coupling loss factors in the matrix $\mathbf{C}$ are proportional to the damping loss factor η. This behaviour has been observed elsewhere [12,14–17]. In general, there will be non-zero direct and indirect coupling loss factors. These depend on the statistics of $\psi_{jj}^{(r)}$ and no simple expressions for them exist. However, limiting cases exist where all the system modes are global and all are local, and here expressions for the coupling loss factors can be found.

3.1.1. All modes global: equipartition of energy

In the limit when all the system modes are global, then the kinetic energy for each mode is distributed uniformly throughout the system and $\psi_{jj}^{(r)} \to v_r$. Consequently $\sigma_r^2 \to 0$, $\rho_{rs} \to 0$ and

$A_{rs} \to \nu_r/(\omega\eta)$. The modal power in the rth subsystem is given by

$$\frac{E_r}{n_r} = \frac{1}{\omega\eta n_{tot}} \sum_s P_{in,s} \tag{23}$$

and is thus the same for all subsystems. This corresponds to equipartition of energy. Equipartition therefore occurs in the low damping, low modal overlap limit, but *only* if all the system modes are global. In this limit the coupling loss factors become indeterminate, since $\mathbf{A}$ is singular.

3.1.2. All modes local: "proper-SEA"

If all the modes are *entirely* local, then $\psi_{jj}^{(r)}$ is 1 or 0, depending on whether or not the jth mode is localized within the rth subsystem. The variances reach limits given by $\sigma_r^2 = \nu_r - \nu_r^2$; $\rho_{rs} = -\nu_r \nu_s$, $\mathbf{A}$ becomes diagonal, with all the power input to a subsystem being dissipated within that subsystem, and the coupling loss factors are zero: the subsystems are uncoupled.

A local mode is now defined to be one which is substantially localized within one subsystem (for which $\psi_{jj}^{(s)} = O(1)$) together with its immediate neighbours to which it is physically connected (for which $\psi_{jj}^{(r)} = O(\varepsilon)$ is small). For the remaining subsystems to which s is not directly coupled $\psi_{jj}^{(r)} \leqslant O(\varepsilon^2)$ is very small. If all the modes are local the elements in $\boldsymbol{\alpha}$ become small and all higher powers of $\boldsymbol{\alpha}$ in Eq. (19) can be neglected so that

$$\mathbf{C} \approx \eta\boldsymbol{\alpha},$$

$$\eta_{rs} \approx \eta \frac{\mathrm{E}[\psi_{jj}^{(r)}\psi_{jj}^{(s)}]}{\nu_r} = \eta \frac{\nu_r \nu_s + \rho_{rs}}{\nu_r}. \tag{24}$$

Thus the coupling loss factors, both direct and indirect, are proportional to η and depend on the modal densities and mode shape covariances ρ_{rs}.

Suppose that all the modes are local. If subsystems r and s are directly coupled then the expectation in Eq. (24) is small ($O(\varepsilon)$), and hence the coupling loss factor η_{rs} is proportional to, but much less than, η. However, if subsystems r and s are not directly coupled then the expectation is very small ($\leqslant O(\varepsilon^2)$), and the indirect coupling loss factors are negligible. Thus $\mathbf{A}^{-1}$ becomes a proper-SEA matrix: it is possible, in principle, to apply proper-SEA in its conventional sense—all indirect coupling loss factors are zero—but with the proviso that the direct coupling loss factors are proportional to the damping loss factor.

3.1.3. A mixture of local and global modes

In the general case there will be some mixture of local and global modes, there may be non-zero indirect coupling loss factors and all coupling loss factors will be proportional to the damping loss factor. Conventional SEA predicts a response determined by equipartition of energy but the actual response differs from this. For the directly excited and unexcited subsystems the responses are in the ratios

$$\left(\frac{E}{E_{equip}}\right)_{ss} = \frac{\mathrm{E}[\psi_{jj}^{(s)2}]}{\nu_s^2} = 1 + \frac{\sigma_s^2}{\nu_s^2},$$

$$\left(\frac{E}{E_{equip}}\right)_{rs} = \frac{\mathrm{E}[\psi_{jj}^{(r)}\psi_{jj}^{(s)}]}{\nu_r \nu_s} = 1 + \frac{\rho_{rs}}{\nu_r \nu_s}, \quad r \neq s. \tag{25}$$

Thus classical SEA always underpredicts the response of the driven system and usually overpredicts the response in undriven subsystems (the covariance ρ_{rs} is generally negative but may be positive in systems with three or more subsystems). This behaviour has been observed before using wave methods and in numerical examples (e.g. Refs. [9–14]).

4. Increasing modal overlap

Under the stated assumptions the energy influence coefficients are given by Eq. (14) with μ_{jk} defined in Eq. (10). Cross-modal terms are important for those mode pairs (j,k) which overlap. For these $\mu_{jk} \approx 1$. As the damping and modal overlap increase, there are more such overlapping pairs of modes.

Eq. (14) can be written in terms of the interaction of mode j with its mth neighbour as

$$A_{rs} = \frac{1}{\omega\eta} \frac{\mathrm{E}\left[\psi_{jj}^{(r)}\psi_{jj}^{(s)}\right] + \mathrm{E}\left[\sum_{m,m\neq 0} \mu_{j,j+m}\psi_{j,j+m}^{(r)}\psi_{j,j+m}^{(s)}\right]}{v_s},$$

$$\mu_{j,j+m} = \frac{M_{j,j+m}^2}{1+M_{j,j+m}^2},$$

$$M_{j,j+m} = \frac{\Delta}{(\omega_{j+m}-\omega_j)} \sim \frac{M}{m}. \tag{26}$$

In the last of these equations it has been noted that $(\omega_{j+m}-\omega_j) \sim m/n_{tot}(\omega)$. The numerator of A_{rs} is the sum of two terms. The first, involving the 'self' modal terms ψ_{jj}, is the $M \to 0$ limit discussed in Section 3. The second term comprises the expectation of cross-modal terms, which are usually negative. If modes j and $j+m$ overlap $\mu_{j,j+m} \approx 1$, otherwise $\mu_{j,j+m} \approx 0$.

In general, the situation is very complicated and little can be said about the coupling loss factors. However, some general observations can be made. As an approximation, non-overlapping mode pairs can be ignored, since for these $\mu_{j,j+m} \approx 0$. If the modal overlap is M, then a particular mode typically overlaps its $\pm M$ neighbours, for which $\mu_{j,j+m} \approx 1$. Thus one can write as a simple approximation

$$A_{rs} \sim \frac{1}{\omega\eta} \frac{\mathrm{E}[\psi_{jj}^{(r)}\psi_{jj}^{(s)}] + \mathrm{E}\left[\sum_{m=-M\; m\neq 0}^{M} \psi_{j,j+m}^{(r)}\psi_{j,j+m}^{(s)}\right]}{v_s}. \tag{27}$$

As the modal overlap M increases, more terms are included in the second expectation. Since these are predominantly negative, $\omega\eta A_{rs}$ (the power dissipated in subsystem r per unit power input to subsystem s) generally decreases with increasing modal overlap (and hence increasing η) if $r \neq s$. However, $\omega\eta A_{ss}$ generally increases with increasing modal overlap. Thus, the dissipated powers become proportionately larger in the excited subsystem and smaller in the indirectly excited subsystems, as one would expect.

Both A_{rs} and the coupling loss factors η_{sr} depend on the detailed nature of the mode shapes, the loss factor and the natural frequencies (through the term μ_{jk}). The coupling loss factors are no longer proportional to η, since μ_{jk} also depends on η. For low to moderate modal overlap, relatively few terms contribute to A_{rs}, and hence larger variability might be expected.

Furthermore, there also tends to be some significant correlation between $\mu_{j,j+m}$ and $\psi_{j,j+m}$ for small m, because of mode repulsion effects (the natural frequencies of two modes with strongly correlated mode shapes tend to repel each other, while modes with close natural frequencies tend to be weakly correlated).

Finally, if all the modes are local, the conclusions drawn in Section 3.1.2 apply here as well. In that subsection a local mode was defined to be one which is large within only one subsystem, small within its immediate neighbours to which it is physically connected and very small within all other subsystems. The dominant terms in the off-diagonal elements of **A** are now of two types. The first are the self-terms $\psi_{jj}^{(r)}\psi_{jj}^{(s)}$, with r and s being physically connected and mode j being local to one of them. The second type arises from those mode pairs j and k for which $\psi_{jk}^{(r)}\psi_{jk}^{(s)}$ is largest. These involve a pair of modes which are both substantially local to either subsystem r or subsystem s. It can be shown that $\psi_{jk}^{(r)}\psi_{jk}^{(s)} \leqslant O(\varepsilon^{3/2})$ for indirectly coupled subsystems. Thus the off-diagonal elements of **A** which are largest correspond to subsystems r and s which are physically connected. When inverted (to determine the coupling loss factor matrix) the largest terms in **C** (the coupling loss factor matrix) thus also relate to subsystems which are physically connected. Therefore, if all modes are local, a proper-SEA model can be formed: indirect coupling loss factors are negligible. However, the coupling loss factors are related to the statistics of the local modes in a complicated manner.

5. High modal overlap

For high modal overlap very many terms contribute to the energy influence coefficient matrix **A**. Eq. (26) can be re-written as

$$A_{rs} = -\frac{1}{\omega\eta}\frac{\mathrm{E}\left[\sum_m (1-\mu_{j,j+m})\psi_{j,j+m}^{(r)}\psi_{j,j+m}^{(s)}\right]}{\nu_s}. \tag{28}$$

This follows by noting that [18]

$$\sum_m \psi_{j,j+m}^{(r)}\psi_{j,j+m}^{(s)} = 0, \quad r \neq s. \tag{29}$$

Eq. (29) follows from modal incoherence. One interpretation of the result follows from the observation that, as $\eta, M \to \infty$, the response becomes more and more localized around the excitation point and decreases ever more rapidly away from the excited region. As a consequence, A_{rs} $(r \neq s)$ decreases as the modal overlap increases, as one would expect: for large damping the response becomes significant only in the directly excited subsystem. If one again uses the approximation $\mu_{j,j+m} \approx 1$ for overlapping mode pairs then

$$A_{rs} \sim \frac{1}{\omega\eta}\frac{-\mathrm{E}\left[\sum_{m,m\notin(-M,M)} \psi_{j,j+m}^{(r)}\psi_{j,j+m}^{(s)}\right]}{\nu_s}, \quad r \neq s \tag{30}$$

where the sum now runs over all modes except those that overlap.

Thus as the modal overlap increases, A_{rs} $(r \neq s)$ becomes smaller compared to the diagonal terms, and the elements of the matrix $\boldsymbol{\alpha}$ in Eq. (17) become smaller. Thus only the first terms in the expansion for **C** need be retained, so that $\mathbf{C} \approx \eta\boldsymbol{\alpha}$, and therefore $\eta_{sr} = -C_{rs} \approx -\eta\alpha_{rs} = \omega\eta^2 A_{rs}$.

Noting that $\nu_s = n_s/n_{tot}$ and $M = \omega\eta n_{tot}$ it follows that

$$\omega n_s \eta_{sr} \sim M\left(-\mathrm{E}\left[\sum_{m, m\notin(-M,M)} \psi^{(r)}_{j,j+m}\psi^{(s)}_{j,j+m}\right]\right). \tag{31}$$

This is defined here to be the weak coupling/high modal overlap limit. The coupling loss factors in Eq. (31) depend on the modal overlap explicitly through the factor M and implicitly through the expectation of the sum.

5.1. Asymptotic dependence of the coupling loss factors: proper SEA

This section concerns the asymptotic behaviour of the coupling loss factors of Eq. (31) in the high modal overlap limit. Strictly the discussion applies to wave-bearing subsystems. It is seen that the indirect coupling loss factors asymptote to zero, so that a proper-SEA model can be formed, while the direct coupling loss factors asymptote to constants.

By definition

$$\psi^{(r)}_{j,j+m} = \int_r \rho(\mathbf{x})\phi_j(\mathbf{x})\phi_{j+m}(\mathbf{x})\,\mathrm{d}\mathbf{x}, \tag{32}$$

where $\mathbf{x}$ is the position vector in subsystem r and the integral is to be evaluated over subsystem r. Now the jth mass-normalized mode shape typically varies in subsystem r as $\sqrt{\rho(\mathbf{x})}\phi_j(\mathbf{x}) \sim \cos(\mathbf{k}^{(r)}_j \cdot \mathbf{x})$, where $\mathbf{k}^{(r)}_j$ is the wavenumber vector. Thus the integrand in Eq. (32) involves the term

$$\rho(\mathbf{x})\phi_j(\mathbf{x})\phi_{j+m}(\mathbf{x}) \sim \cos(\mathbf{k}^{(r)}_j\cdot\mathbf{x})\cos(\mathbf{k}^{(r)}_{j+m}\cdot\mathbf{x}) = \tfrac{1}{2}(\cos((\mathbf{k}^{(r)}_{j+m} - \mathbf{k}^{(r)}_j)\cdot\mathbf{x}) + \cos((\mathbf{k}^{(r)}_{j+m} + \mathbf{k}^{(r)}_j)\cdot\mathbf{x})). \tag{33}$$

The terms on the right hand side are highly oscillatory unless the wavenumber vectors $\mathbf{k}^{(r)}_j$ and $\mathbf{k}^{(r)}_{j+m}$ have a similar direction. In this case the first term is oscillatory, but not highly oscillatory. When integrated over subsystem r, the orders of magnitudes of the two terms on the right hand side of Eq. (33) are $O((k^{(r)}_{j+m} - k^{(r)}_j)^{-1})$ and $O((k^{(r)}_{j+m} + k^{(r)}_j)^{-1})$, respectively, so that the first term is dominant. Now

$$k^{(r)}_{j+m} - k^{(r)}_j \approx \frac{(\omega_{j+m} - \omega_j)}{c^{(r)}_g} \sim \frac{m}{n_{tot}c^{(r)}_g}, \tag{34}$$

where $c^{(r)}_g$ is the group velocity. Hence it follows that

$$\psi^{(r)}_{j,j+m} = O(m^{-1}). \tag{35}$$

Thus both $\psi^{(r)}_{j,j+m}$ and $\psi^{(s)}_{j,j+m}$ in Eq. (31) are $O(m^{-1})$, so that their product is $O(m^{-2})$. However, integrating Eq. (33) gives sinusoidal terms, so that the signs of $\psi^{(r)}_{j,j+m}$ and $\psi^{(s)}_{j,j+m}$, and the sign of their product, may be positive or negative. Noting that

$$\sum_{M+1}^{\infty} \frac{1}{m^2} = \Psi(M+1) = \frac{1}{(M+1)} + \frac{1}{2(M+1)^2} + \cdots, \tag{36}$$

where $\Psi(M)$ is the Digamma function, it follows that

$$\sum_{m,m\notin(-M,M)} \psi^{(r)}_{j,j+m}\psi^{(s)}_{j,j+m} \leqslant \sum_{m,m\notin(-M,M)} |\psi^{(r)}_{j,j+m}\psi^{(s)}_{j,j+m}| = O(M^{-1}). \tag{37}$$

From Eq. (31) it therefore follows that

$$\omega n_s \eta_{sr} \leqslant O(M^0). \tag{38}$$

This indicates that, in the high modal overlap limit the coupling loss factors all tend to constants as the modal overlap increases, but the constants may be zero.

Consider now two subsystems which are not directly coupled. The terms in the sum and expectation in Eq. (31) are the products of $\psi^{(r)}_{j,j+m}$ and $\psi^{(s)}_{j,j+m}$, which are found by integrating Eq. (33), and each of which may be positive or negative. The dominant term in the product $\psi^{(r)}_{j,j+m}\psi^{(s)}_{j,j+m}$ will be of the form

$$\sin((\mathbf{k}^{(r)}_{j+m} - \mathbf{k}^{(r)}_{j}) \cdot \mathbf{x}_r) \sin((\mathbf{k}^{(s)}_{j+m} - \mathbf{k}^{(s)}_{j}) \cdot \mathbf{x}_s) \tag{39}$$

and is evaluated at the boundary locations $\mathbf{x}_r$ and $\mathbf{x}_s$ of subsystems r and s. Since these positions are distant from each other, the two sinusoidal terms are uncorrelated and their product will be positive or negative. Hence it might be expected that the sum and expectation in Eq. (31) would tend to zero, so that the indirect coupling loss factors are such that $\omega n_s \eta_{sr} < O(M^0)$. Thus the indirect coupling loss factors tend to zero in the weak coupling, high modal overlap limit. This results from the fact that the mode shapes at two different, distant points in a structure are uncorrelated.

However, for two subsystems which *are* directly coupled, some of the terms like Eq. (39) will involve points lying on the coupling between subsystems r and s. For these $\mathbf{x}_r = \mathbf{x}_s$, so that while the individual sinusoidal terms may be positive or negative, they have the same sign, so that their product is always positive—the mode shapes in the rth and sth subsystems at their common boundary are correlated. Thus, the sum and expectation of these positive terms is a positive constant, so that the direct coupling loss factors $\omega n_s \eta_{sr} = O(M^0)$, and therefore asymptote to positive constants, independent of modal overlap.

In summary, as the modal overlap increases, the indirect coupling loss factors tend to zero, while the direct coupling loss factors tend to positive constants. Therefore the behaviour of the system can be described by a proper-SEA model. This behaviour occurs because of mode shape correlation at the boundary between directly coupled subsystems. While the discussion above was developed for wave-bearing subsystems, the conclusions regarding the correlation of mode shapes between directly and indirectly coupled subsystems would be expected to apply in more general circumstances.

6. System comprising two subsystems

Consider the special case of a system comprising just two subsystems. Clearly, all modes are local in the sense defined earlier and there can be no indirect coupling loss factors, so a proper-SEA model can always be formed if the bandwidth of excitation is large enough. The cross-mode participation factors are such that $\psi_{jk}^{(1)} + \psi_{jk}^{(2)} = \delta_{jk}$, while $v_1 + v_2 = 1$.

6.1. Low modal overlap

The variances and covariances in Eq. (22) are now such that

$$\sigma_1^2 = \sigma_2^2 = -\rho_{12} = \sigma^2, \\ 0 \leqslant \sigma^2 \leqslant v_1 v_2. \tag{40}$$

The limits on σ^2 correspond, respectively, to the cases of all modes being global and all being local. The coupling loss factor becomes

$$\omega n_1 \eta_{12} = M \frac{v_1 v_2 (v_1 v_2 - \sigma^2)}{\sigma^2}. \tag{41}$$

Proper-SEA can be applied (as long as there are enough modes in the band), but the coupling loss factors depend on the mode shape statistics through the means and the variance and are proportional to the damping loss factor.

6.2. Higher modal overlap

At higher modal overlap cross-mode shape statistics become important. Since $\psi_{j,j+m}^{(2)} = -\psi_{j,j+m}^{(1)}$ it follows that the covariances $\rho_{rs,m}$ of $\psi_{j,j+m}^{(r)} \psi_{j,j+m}^{(s)}$ are related by $\sigma_{11,m}^2 = \sigma_{22,m}^2 = -\rho_{12,m} = \sigma_{(m)}^2$. $\mathbf{A}$ can be written as

$$\mathbf{A} = \frac{1}{\omega\eta} \begin{bmatrix} \dfrac{v_1^2 + (\sigma^2 + S)}{v_1} & \dfrac{v_1 v_2 - (\sigma^2 + S)}{v_2} \\ \dfrac{v_1 v_2 - (\sigma^2 + S)}{v_1} & \dfrac{v_2^2 + (\sigma^2 + S)}{v_2} \end{bmatrix}, \\ S = \mathrm{E}\left[\sum_{m, m \neq 0} \mu_{j,j+m} \, \psi_{j,j+m}^{(1)^2} \right] \tag{42}$$

with the sum S increasing from zero as the modal overlap increases. Consequently,

$$\omega n_1 \eta_{12} = M \frac{v_1 v_2 (v_1 v_2 - (\sigma^2 + S))}{(\sigma^2 + S)}. \tag{43}$$

For large modal overlap $S \sim v_1 v_2 - \sigma^2 - \mathrm{E}\,[\sum_{m, m \notin (-M,M)} \psi_{j,j+m}^{(1)^2}]$ and hence in the weak coupling limit the coupling loss factor is

$$\omega n_1 \eta_{12} \sim M \mathrm{E}\left[\sum_{m, m \notin (-M,M)} \psi_{j,j+m}^{(1)^2} \right]. \tag{44}$$

7. Examples

7.1. Uniform rod

Consider the system comprising a single, uniform rod undergoing axial vibration as shown in Fig. 1. The rod lies along the x-axis, is of length L and the ends are fixed. It is divided into an arbitrary number of subsystems, with the rth subsystem being the region from x_{r-1} to x_r. This is of course a somewhat artificial example: one would not choose to model this system using SEA, and large modal overlap corresponds to large wave decay rates because the system is one dimensional. However, it is the simplest of examples to illustrate the foregoing text: qanalytical expressions for the mode shapes exist, the modal density is constant and the system is non-dispersive.

For the jth mode

$$\omega_j = j\left(\frac{\pi}{L}\sqrt{\frac{E}{\rho}}\right),$$
$$\phi_j(x) = \sqrt{\frac{2}{\rho AL}}\sin\frac{j\pi x}{L}, \tag{45}$$

where E, ρ and A are the elastic modulus, density and cross-sectional area. The modal density is

$$n_{tot}(\omega) = \frac{L}{\pi}\sqrt{\frac{\rho}{E}}. \tag{46}$$

The natural frequencies are uniformly spaced in frequency. All the directly coupled subsystems are strongly connected, since the power transmission coefficient $\tau = 1$ for all the couplings.

For the rth subsystem

$$\psi_{jj}^{(r)} = \left(\frac{x_r - x_{r-1}}{L}\right) - \frac{1}{2j\pi}\left(\sin\frac{2j\pi x_r}{L} - \sin\frac{2j\pi x_{r-1}}{L}\right),$$
$$\psi_{j,j+m}^{(r)} = \frac{1}{\pi}\left[\frac{1}{m}\left(\sin\frac{m\pi x_r}{L} - \sin\frac{m\pi x_{r-1}}{L}\right)\right.$$
$$\left. - \frac{1}{(2j+m)}\left(\sin\frac{(2j+m)\pi x_r}{L} - \sin\frac{(2j+m)\pi x_{r-1}}{L}\right)\right]. \tag{47}$$

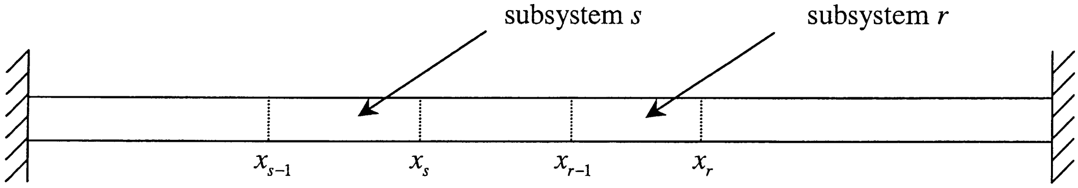

Fig. 1. System comprising a single rod.

The frequency average and expected values of the modal energy participation factor $\psi_{jj}^{(r)}$ are, for large enough j or for a wide enough band,

$$\overline{\psi_{jj}^{(r)}} = \mathrm{E}[\psi_{jj}^{(r)}] = \frac{(x_r - x_{r-1})}{L} = v_r. \tag{48}$$

All modes are global, since $\psi_{jj}^{(r)} \approx v_r$ for all j. For higher values of j (and small or moderate m) only the first terms in Eq. (47) become important, since $2j$ is typically large. Furthermore, the second terms are highly oscillatory with increasing j and m. Retaining only the first terms gives

$$\psi_{jj}^{(r)} \approx \left(\frac{x_r - x_{r-1}}{L}\right),$$
$$\psi_{j,j+m}^{(r)} \approx \frac{1}{m\pi}\left(\sin\frac{m\pi x_r}{L} - \sin\frac{m\pi x_{r-1}}{L}\right). \tag{49}$$

Suppose $x_s \leqslant x_{r-1}$ (i.e., subsystem s lies to the left of subsystem r). Using the approximations of Eq. (49), the term $\psi_{j,j+m}^{(r)}\psi_{j,j+m}^{(s)}$ can be written as

$$\begin{aligned}\psi_{j,j+m}^{(r)}\psi_{j,j+m}^{(s)} = \frac{1}{m^2}\frac{1}{2\pi^2}\Bigg[&-\cos\left(\frac{m\pi(x_{r-1}-x_s)}{L}\right)\\ &+\left(\cos\left(\frac{m\pi(x_r-x_s)}{L}\right)+\cos\left(\frac{m\pi(x_{r-1}-x_{s-1})}{L}\right)-\cos\left(\frac{m\pi(x_r-x_{s-1})}{L}\right)\right)\\ &+\left(\cos\left(\frac{m\pi(x_{r-1}+x_s)}{L}\right)-\cos\left(\frac{m\pi(x_r+x_s)}{L}\right)-\cos\left(\frac{m\pi(x_{r-1}+x_{s-1})}{L}\right)\right.\\ &\left.+\cos\left(\frac{m\pi(x_r+x_{s-1})}{L}\right)\right)\Bigg].\end{aligned} \tag{50}$$

The product thus decreases as m^{-2}, as was inferred for the general case in Section 5.1. The terms in brackets have been divided into three groups. The first comprises the single term involving $(x_{r-1} - x_s)$, and will equal -1 if subsystems r and s are directly coupled (i.e., if $x_s = x_{r-1}$), otherwise it will be oscillatory as m increases. The second group comprises three terms involving co-ordinate differences: these are oscillatory as m increases, but may be slowly oscillatory if subsystems r and s are close. The third group involves four oscillatory terms involving co-ordinate sums which are generally highly oscillatory. Thus if subsystems r and s are directly coupled

$$\psi_{j,j+m}^{(r)}\psi_{j,j+m}^{(s)} = -\frac{1}{m^2}\frac{1}{2\pi^2} + \text{oscillatory terms} \tag{51}$$

while if they are not coupled, all the terms are oscillatory.

For this case $\omega_{j+m} - \omega_j = m/n_{tot}$ exactly. Thus in Eq. (26) we can write

$$\mu_{j,j+m} = \frac{1}{1+(m^2/M^2)}. \tag{52}$$

At low modal overlap, $M \ll 1$, only the terms $\psi_{jj}^{(r)}$ are relevant and equipartition of energy occurs because the modes of the system are global. As the modal overlap increases, the interactions of neighbouring modes start to become important. Each of these interactions gives a contribution of the form $\mu_{j,j+m}\psi_{j,j+m}^{(r)}\psi_{j,j+m}^{(s)}$ (Eq. (26)) where $\mu_{j,j+m}$ is only significant if the modes overlap.

For low-to-moderate M, a few terms in Eq. (26) become significant, typically only those for which $|m| \leqslant M$. The leading terms (the first group in Eq. (50)) are negative, so the off-diagonal energy influence coefficients A_{rs} tend to decrease. If $M \sim 1$ then A_{rs} will depend strongly on the subsystem co-ordinates x.

For moderate-to-high M, the energy influence coefficients can be written as (Eq. (28))

$$A_{rs} = -\frac{1}{\omega\eta}\frac{\mathrm{E}\left[\sum_m (1-\mu_{j,j+m})\psi^{(r)}_{j,j+m}\psi^{(s)}_{j,j+m}\right]}{v_s}. \tag{53}$$

If subsystems r and s are not coupled $\psi^{(r)}_{j,j+m}\psi^{(s)}_{j,j+m}$ only involves oscillatory terms and the expectation and sum tend to zero. However, if they are coupled, A_{rs} can be evaluated using the expressions in Eqs. (51) and (52). Reversing the order of summation and expectation, and assuming that the expectation of the oscillatory terms in $\psi^{(r)}_{j,j+m}\psi^{(s)}_{j,j+m}$ is zero, it follows that

$$\begin{aligned} A_{rs} &= -\frac{1}{\omega\eta v_s}\sum_m\left(1-\frac{1}{1+(m^2/M^2)}\right)\left(-\frac{1}{m^2}\frac{1}{2\pi^2}\right) \\ &= \frac{1}{\omega\eta v_s}\frac{1}{2\pi^2}\frac{1}{M^2}\sum_m\frac{1}{1+(m^2/M^2)}. \end{aligned} \tag{54}$$

Now $\sum_{-\infty}^{\infty} 1/(1+(m^2/M^2)) = \pi M\coth(\pi M)$ and hence, if the modal overlap and the centre frequency (or mode number j) are high enough then, for coupled subsystems,

$$A_{rs} = \frac{1}{\omega\eta v_s}\frac{1}{2\pi}\frac{1}{M}. \tag{55}$$

As the modal overlap increases, the off-diagonal elements A_{rs} $(r \neq s)$ becomes smaller compared to the diagonal terms. The matrix **A** thus consists of relatively large diagonal entries, relatively small entries on the super- and sub-diagonals (i.e., relating coupled subsystems) and zeros elsewhere. The indirect coupling loss factors thus tend to zero. Only the first terms in the expansion for **C** in Eq. (17) need be retained and therefore $\eta_{sr} \approx \omega\eta^2 A_{rs}$. The direct coupling loss factor is therefore given by

$$\omega n_s \eta_{sr} = \frac{1}{2\pi}. \tag{56}$$

This is the same value as the conventional wave estimate of the coupling loss factor given by $\eta_{sr} = \tau/2\pi n_s\omega$ [2], since the power transmission coefficient $\tau = 1$.

This behaviour can be interpreted in terms of the correlation of the cross-modal interactions. If two subsystems are not directly coupled, the system mode shapes in the two subsystems are, on average, uncorrelated—hence the oscillatory terms in Eq. (50). The average interaction thus tends to zero as an increasing number of mode pairs interact (i.e., as the modal overlap increases). However, if the subsystems are directly coupled so that $x_s = x_{r-1}$ then the mode shapes are uncorrelated within the interiors of the subsystems, but *not at the junction between them*—the first term in Eq. (50) now becomes $1/2m^2\pi^2$, is not oscillatory and does not average to zero.

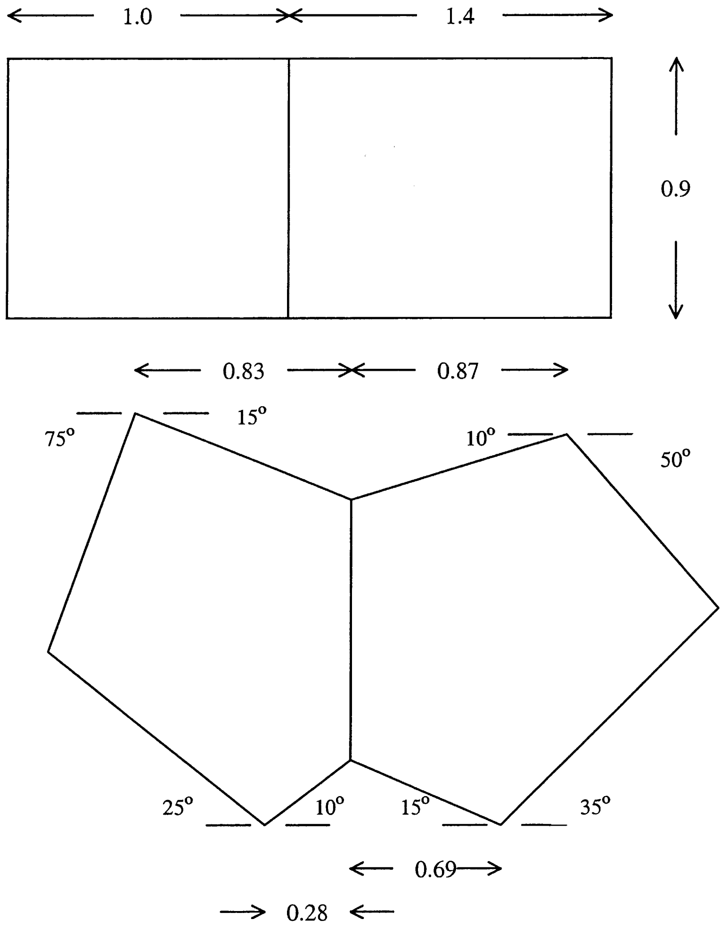

Fig. 2. Systems comprising two coupled rectangular (RR) or pentagonal (PP) plates.

7.2. Two coupled plates

A system comprising two coupled plates has been used as an example in many previous studies. There are just two subsystems, as described in Section 6, and hence there can be no indirect coupling loss factors. In a conventional SEA model the asymptotic modal density and the coupling loss factor are given by

$$n(\omega) = \frac{A}{4\pi}\sqrt{\frac{\rho h}{D}},$$
$$\eta_{12} = \frac{L k_1 \langle \tau_{12} \rangle}{2\pi^2 \omega n_1(\omega)}, \tag{57}$$

where h and D are the plate thickness and bending stiffness and L is the length of the coupled side. A slightly different expression for the modal density can be used which involves corrections for the perimeter and number of corners [2,18].

Here, two different 2-plate systems, taken from Ref. [14], will be considered. In Ref. [14] FE studies were conducted on systems comprising two plates of various shapes, coupled along one edge, and with all edges, and the line of coupling, being simply supported. The length of the coupled edge, the plate areas and all other plate properties were held constant, the aim being to investigate the importance of irregularity in the shape of the plates on the response. More details, including system properties, can be found in Ref. [14]. Results are shown here for two coupled rectangular plates (the "RR" system) and for two coupled pentagonal plates (PP) as shown in Fig. 2. The RR system is regular, and an analytical expression exists for the ensemble averaged coupling loss factor [11]. The PP system is irregular. Conventional SEA treats these systems in the same way. However, their mode shapes, and the mode shape statistics, are different, and this leads to different behaviour, especially at low and moderate modal overlap.

7.2.1. *Low modal overlap: statistics of* $\psi_{jj}^{(1)}$

At low modal overlap only the self-terms $\psi_{jj}^{(1)}$ are relevant. Fig. 3 shows $\psi_{jj}^{(1)} = 1 - \psi_{jj}^{(2)}$ as a function of the natural frequency f_n for the first 120 modes. There is a clear tendency for the modes of the RR system to be more localised in one or other of the plates, with $\psi_{jj}^{(1)}$ tending to be closer to 0 or 1. The statistics of the mode participation factors are somewhat non-stationary with frequency, especially for the lower modes. Averaging over 100 modes from the 21st gives the statistics shown in Table 1. The frequency average $\overline{\psi_{jj}^{(r)}}$ is very close to the fractional modal density ν_1 for both RR and PP systems. However, the variance for the RR system is by far the larger. For low modal overlap the coupling loss factor of Eq. (41) is proportional to the modal overlap and is substantially larger for the PP system, which has the smaller variance (Table 1).

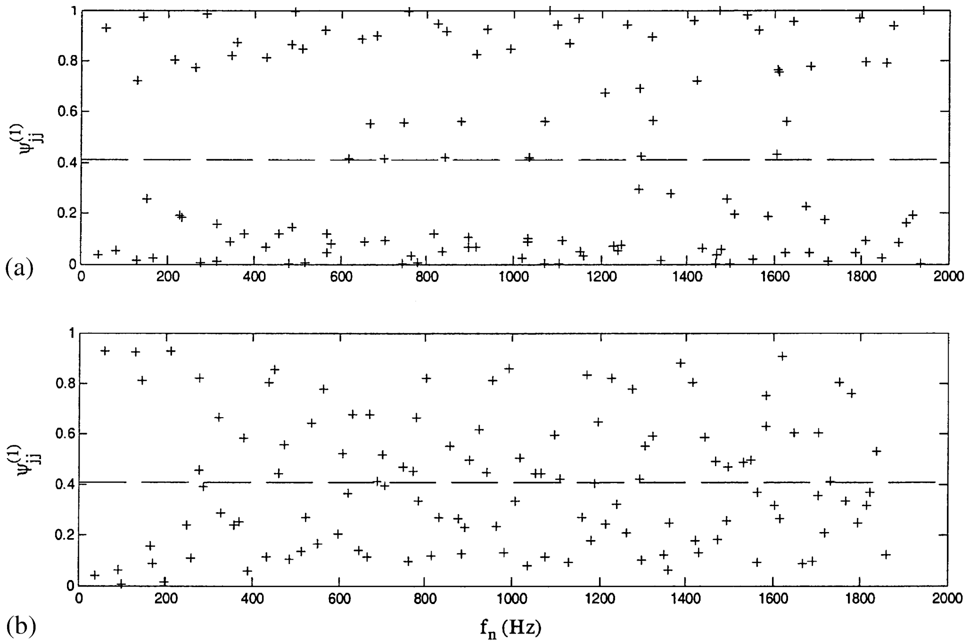

Fig. 3. Self terms $\psi_{jj}^{(1)}$ for (a) RR and (b) PP plate systems and (----) $\overline{\psi_{jj}^{(1)}}$.

Table 1
Mode shape statistics for the "self"-terms of the 2-plate systems

	RR	PP
v_1	0.417	0.417
$\overline{\psi_{jj}^{(1)}}$	0.412	0.410
σ^2	0.143	0.0583
$\omega n_1 \eta_{12}$ at low M	0.169 M	0.771 M

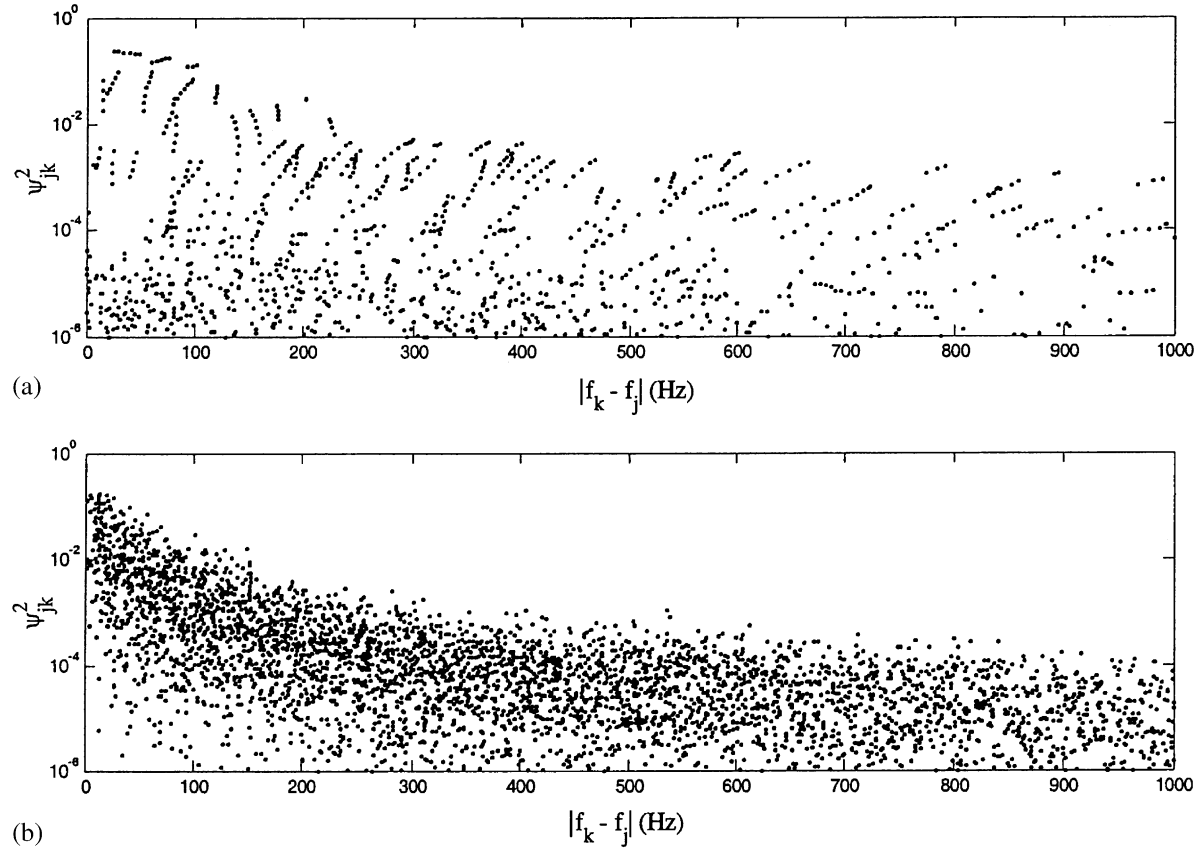

Fig. 4. Cross-mode terms $\psi_{jj}^{(1)^2}$ as a function of natural frequency spacing for (a) RR and (b) PP plate systems.

7.2.2. Higher modal overlap: statistics of $\psi_{jk}^{(1)}$

Fig. 4 shows $\psi_{jk}^{(1)^2} = \psi_{jk}^{(2)^2}$ as a function of the natural frequency spacing of modes j and k. Increasing modal overlap means that more mode pairs are included in the sum S in Eqs. (42) and (43). The values of $\psi_{jk}^{(1)^2}$ appear more random-like for the PP system than for the RR system, for which some clear structure is observable. While there appear to be fewer modes for the RR system, this is an illusion, since many mode pairs are so weakly correlated that $\psi_{jk}^{(1)^2}$ is very small (in theory they have zero correlation, but in FE studies this is not necessarily the case). Also, for a large difference in the natural frequencies, $\psi_{jk}^{(1)^2}$ can be substantially larger for the RR system.

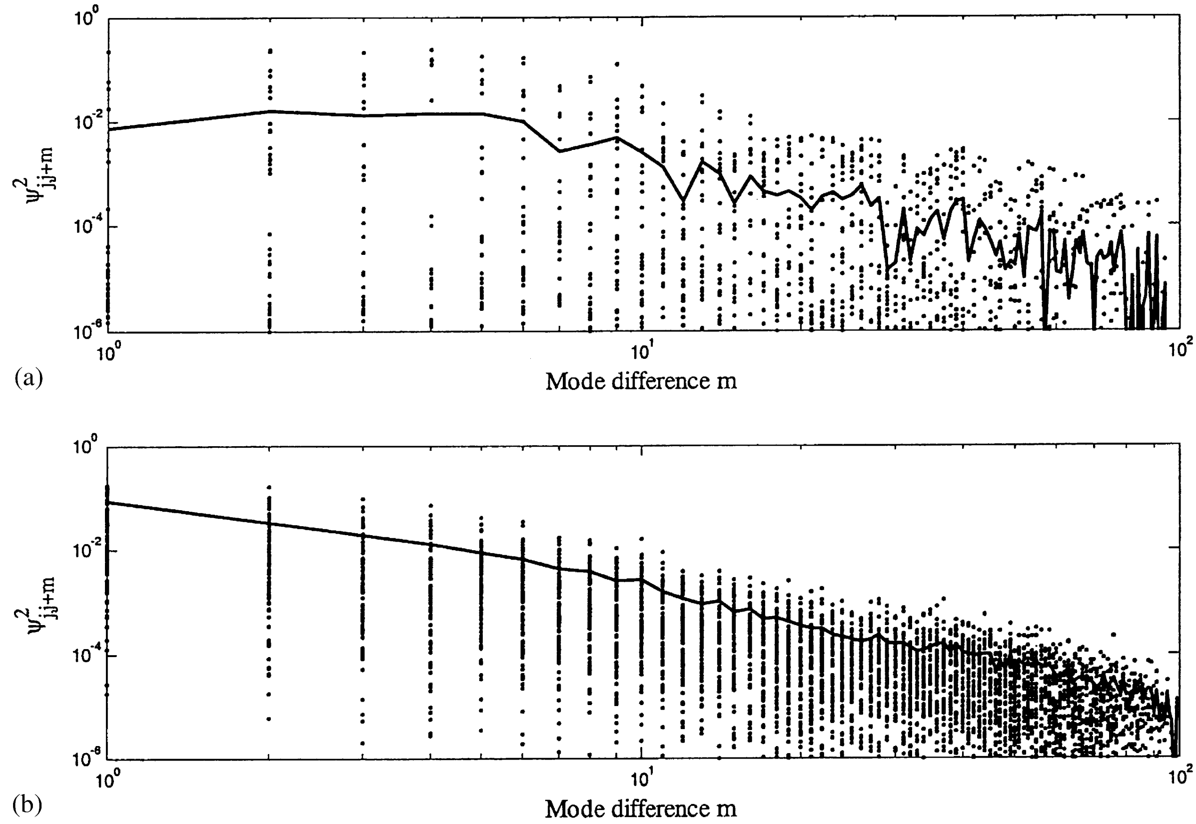

Fig. 5. Cross-mode terms $\psi_{j,j+m}^{(1)^2}$ as a function of mode number difference m for (a) RR and (b) PP plate systems and (—) $E[\psi_{j,j+m}^{(1)^2}]$.

These features are due to the fact that, for the RR system, there are a discrete number of half-wavelengths across the plates for each mode shape. Two modes with different numbers of half-wavelengths are spatially uncorrelated: two modes with the same number of half-wavelengths are well correlated, but tend to have natural frequencies which are well separated.

For higher natural frequency spacing $\psi_{jk}^{(1)^2}$ tends to decrease as $|f_k - f_j|^2$, as expected from the discussion in Section 5.1. This is perhaps more evident in Fig. 5, where the dependence on mode number difference is shown. This last figure also shows the average $E[\psi_{j,j+m}^{(1)^2}]$ as a function of m.

7.2.3. Higher modal overlap: coupling loss factors

Fig. 6 shows the coupling loss factors, normalized with respect to the conventional wave estimate of Eq. (57), as a function of modal overlap. They are found from the energy influence coefficients given by the exact expression (Eq. (6)) and the approximation of Eq. (14) for a frequency band centred at 1 kHz and containing about 27 modes. The low M asymptotes of Eqs. (41) are also shown. There are clear differences between the RR and PP systems at low and moderate M, but the high M asymptotes are very similar. The behaviour is similar for different excitation frequency bands. Differences between the various predictions can be attributed to the effects of out-of-band and off-resonant modes and the fact that the actual excitation only covers a

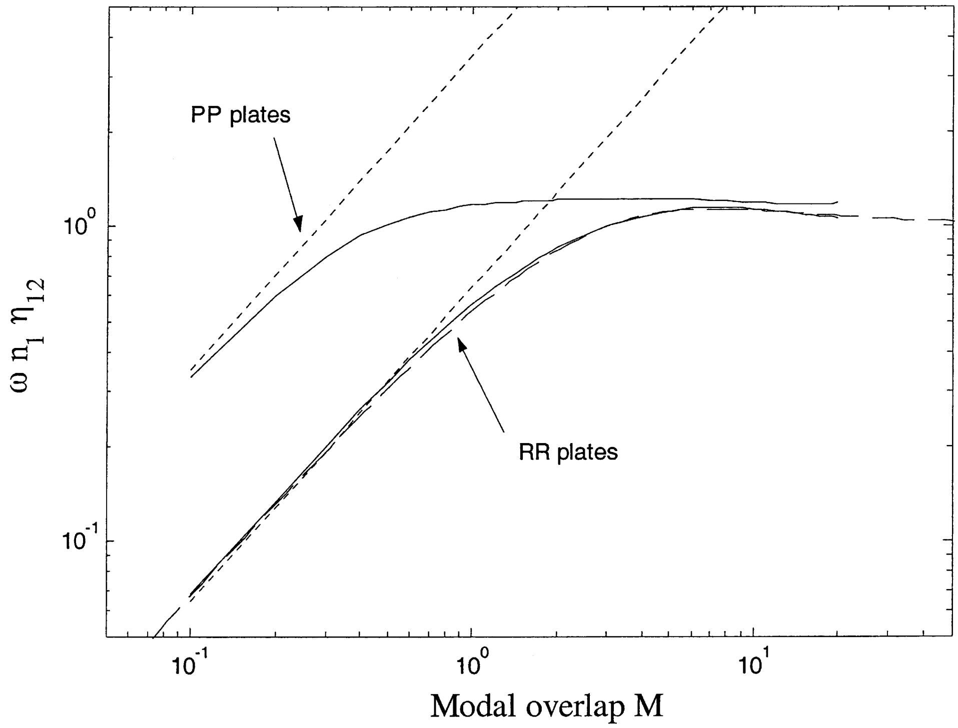

Fig. 6. Coupling loss factor (normalized with respect to conventional estimate of Eq. (67)) as a function of M: (—) PP and RR plate systems; (.....) low M asymptotes; (_ _ _) theoretical expression [16].

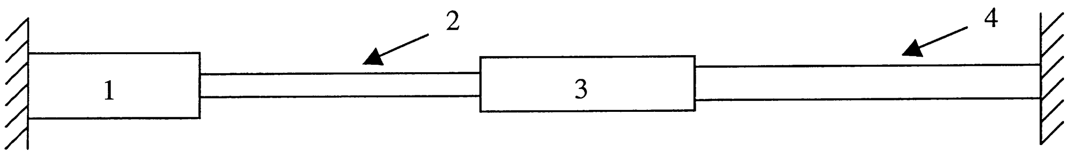

Fig. 7. System comprising 4 coupled rods.

finite frequency band. Furthermore, the simple expressions for the asymptotic modal densities (Eq. (57)) have been used, with no perimeter or corner corrections.

7.3. Four coupled rods

As a final example consider the system comprising 4 rods shown in Fig. 7 undergoing axial vibration. This system can have non-zero indirect coupling loss factors. The properties of the system and the coupling loss factors can be found in Ref. [9]. Here numerical examples of $\psi^{(r)}_{j,j+m}\psi^{(s)}_{j,j+m}$ are presented for 51 modes centred on the 50th natural frequency (i.e., $25 \leqslant j \leqslant 75$), with subsystem $s = 4$ being excited. Interactions with up to 24 nearest neighbours are found (i.e., $-24 \leqslant m \leqslant 24$).

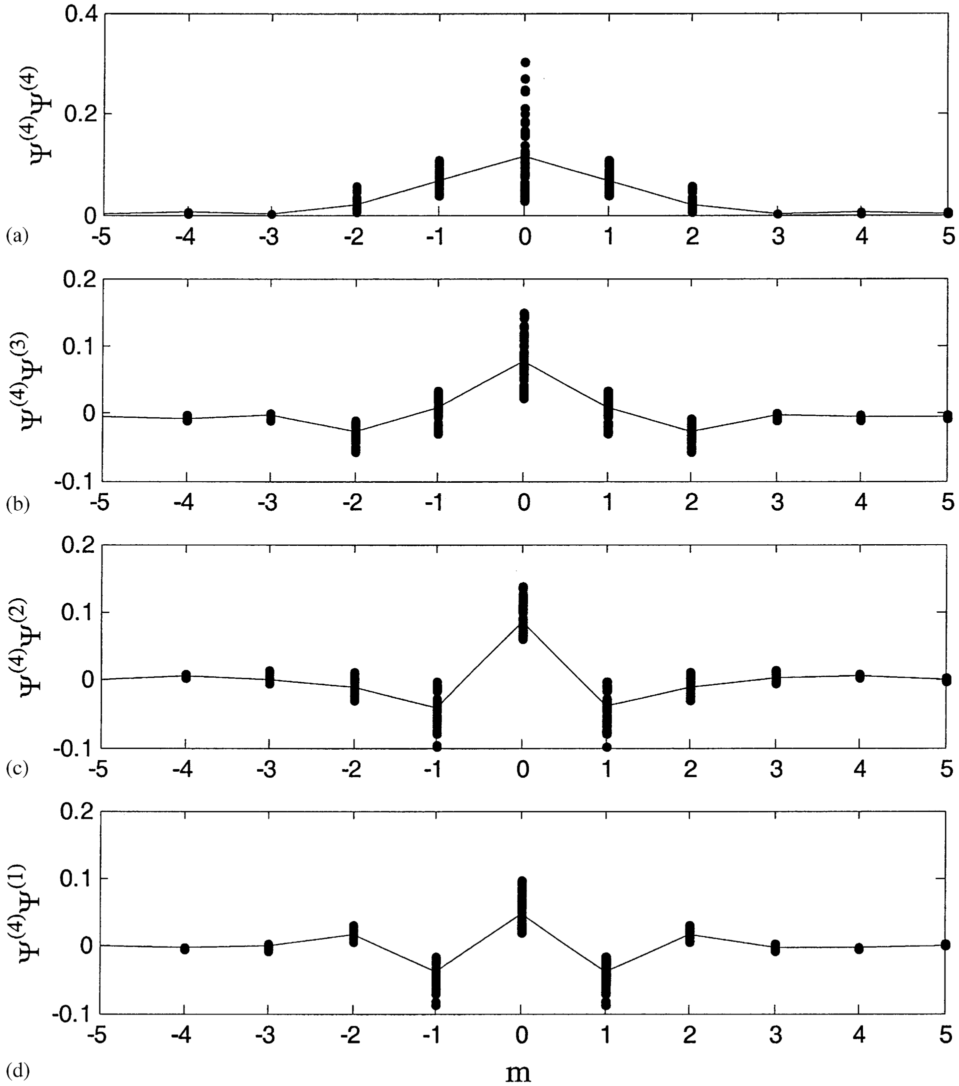

Fig. 8. The terms $\psi^{(r)}_{j,j+m}\psi^{(4)}_{j,j+m}$ for 51 modes centred at the 50th mode of the 4-rod system and (—) the average $\mathrm{E}[\psi^{(r)}_{j,j+m}\psi^{(4)}_{j,j+m}]$ for these modes: (a) $r=4$; (b) $r=3$; (c) $r=2$; (d) $r=1$.

Fig. 8 shows $\psi^{(r)}_{j,j+m}\psi^{(4)}_{j,j+m}$ for all 51 modes in the excitation band. Also shown are the averages $\mathrm{E}[\psi^{(r)}_{j,j+m}\psi^{(4)}_{j,j+m}]$ taken over these modes. Fig. 9 shows the cumulative sums $\sum_{-m}^{m}\mathrm{E}[\psi^{(r)}_{j,j+m}\psi^{(4)}_{j,j+m}]$ as functions of $|m|$. In broad terms, it is these cumulative sums that determine how the energy influence coefficients, and hence the coupling loss factors, vary with increasing modal overlap, since Eq. (27) can equally be written as

$$A_{rs}\sim\frac{1}{\omega\eta}\frac{\mathrm{E}[\psi^{(r)}_{jj}\psi^{(s)}_{jj}]+\sum_{m=-M\ \ m\neq 0}^{M}\mathrm{E}[\psi^{(r)}_{j,j+m}\psi^{(s)}_{j,j+m}]}{\nu_s}. \tag{58}$$

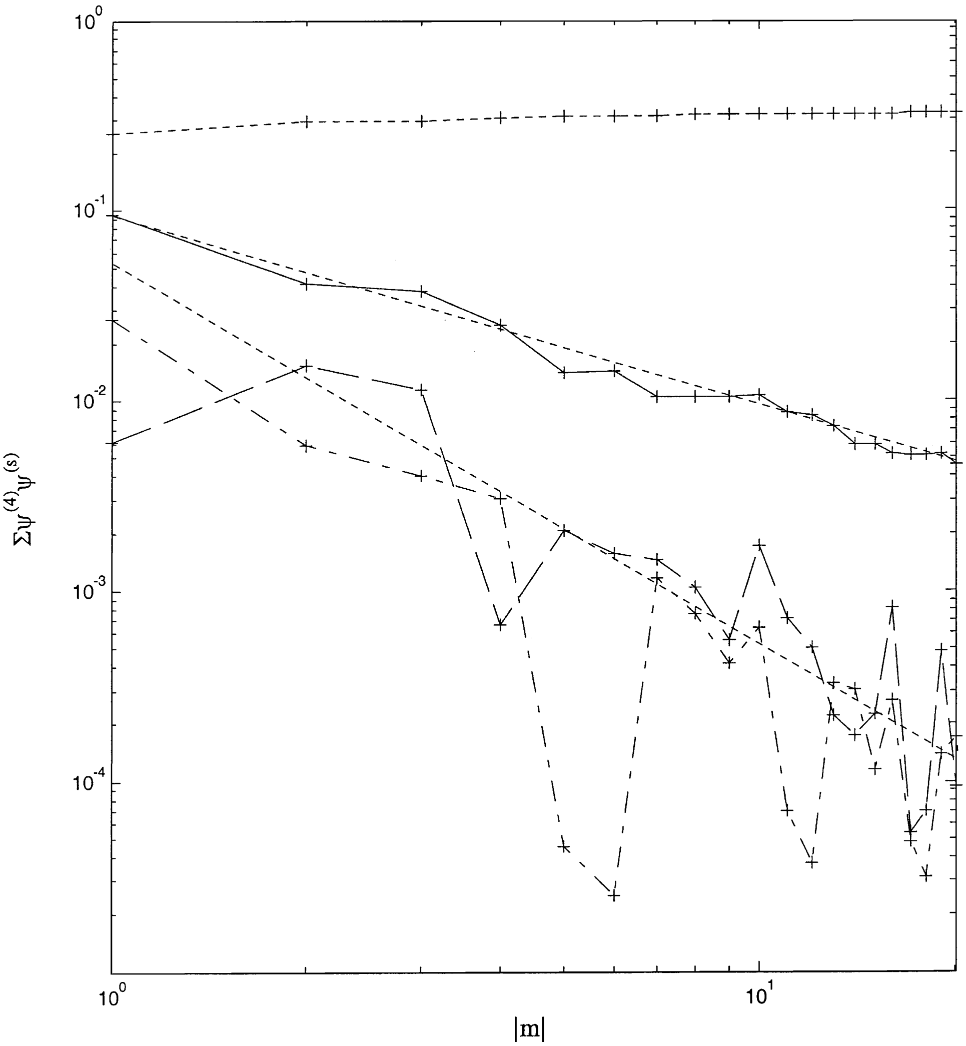

Fig. 9. The cumulative sums $\sum_{-m}^{m} \mathrm{E}[\psi^{(r)}_{j,j+m}\psi^{(4)}_{j,j+m}]$ as functions of $|m|$ for the 4-rod system: (- - + - -) $r = 4$; (– + –) $r = 3$; (— + —) $r = 2$; (– - + - –) $r = 1$. Also shown (– – – –) are lines of slope m^{-1} and m^{-2}.

The self-terms $\psi^{(r)}_{j,j}\psi^{(4)}_{j,j}$ in Fig. 8 are necessarily positive and so too are all cross-terms $\psi^{(4)}_{j,j+m}\psi^{(4)}_{j,j+m}$ for the excited subsystem (Fig. 8(a)). The individual terms and the averages generally decrease with increasing $|m|$. For $r \neq s$ and $m \neq 0$ the individual terms and the averages may be negative. For $|m| = 1$ the terms for the indirectly coupled subsystems are negative (Figs. 8(c) and (d)), with the relative magnitude of the negative terms compared to the $m = 0$ terms being larger for the more distant subsystem (Fig. 8(d)). The cumulative sum can then be expected to decrease somewhat quicker in this case. These features can be explained in terms of typical mode shapes, which involve sinusoidal displacements in the rods. For larger mode difference $|m|$ the subsystem participation factor $\psi^{(r)}_{j,j+m}$ tends to decrease because the difference between the wavenumbers associated with the two modes becomes larger, while for subsystems which are more distantly separated (e.g., $r = 1$ rather than $r = 2$) the product $\psi^{(r)}_{j,j+m}\psi^{(4)}_{j,j+m}$ tends to be more oscillatory with increasing m.

The cumulative sums are shown in Fig. 9 as functions of $|m|$. For the excited system every term $\mathrm{E}[\psi^{(4)}_{j,j+m}\psi^{(4)}_{j,j+m}]$ is positive and the sum asymptotes to a constant equal to the fractional modal density of subsystem 4. For the immediate neighbour, subsystem 3, the cumulative sum $\sum_{-m}^{m}\mathrm{E}[\psi^{(3)}_{j,j+m}\psi^{(4)}_{j,j+m}]$ decreases with increasing $|m|$, but generally decreases as m^{-1} because of the correlation of the mode shapes at the junction between subsystems 4 and 3, as discussed in Section 5.1. The coupling loss factor thus asymptotes to a constant. For the indirectly coupled subsystems (2 and 1) $\sum_{-m}^{m}\mathrm{E}[\psi^{(2,1)}_{j,j+m}\psi^{(4)}_{j,j+m}]$ decreases more rapidly than m^{-1}, as expected, because there is no such correlation of mode shapes. In general, the cumulative sums seem to decrease as m^{-2}, which might be expected: if the signs of successive terms in the cumulative sum are different, then the sum (see Eq. (36)) would be of terms like $(-1)^m/m^2$. The sum of such terms is $\sum_{m}^{\infty}(-1)^m/m^2 = O(m^{-2})$. Thus with increasing modal overlap the cumulative sum $\sum_{-m}^{m}\mathrm{E}[\psi^{(r)}_{j,j+m}\psi^{(4)}_{j,j+m}]$ for the indirectly coupled subsystems decreases rapidly enough such that the indirect coupling loss factors tend to zero.

8. Concluding remarks

This paper concerned the existence of "proper-SEA" and "quasi-SEA" models, the dependence of the direct and indirect coupling loss factors on modal overlap and the conditions under which the indirect coupling loss factors in an SEA model are zero. The results of Ref. [9] were used to express the matrix **A** of energy influence coefficients in terms of the modes of the system. It was assumed that the system is linear and that the excitations applied to the different subsystems are uncorrelated. The excitation is "rain-on-the-roof", defined to be random, spatially delta-correlated and with a spectral density that is independent of frequency, but varies spatially in proportion to the local mass density. The damping is proportional, with all modes having the same bandwidth $\Delta = \omega\eta$. Finally the time-average kinetic and potential energies are assumed equal. In Ref. [9] it was seen that if the bandwidth of the excitation is wide enough so that the modes within it satisfy Eq. (12) then the inverse $\mathbf{A}^{-1}$ is an SEA-like matrix **L**, which thus satisfies the necessary conditions given in Section 1.1. In particular, the coupling loss factors satisfy the consistency relation, Eq. (4). In general, however, **L** is a quasi-SEA matrix with there being non-zero indirect coupling loss factors.

The conditions under which **L** becomes a proper-SEA matrix (i.e., one for which all indirect coupling loss factors are zero) were then investigated, together with the dependence of the coupling loss factors on the modal properties of the system and the modal overlap. These parameters depend on the frequency integrals Γ_{jk} and the cross-mode participation factors $\psi^{(r)}_{jk}$.

In summary, all the indirect coupling loss factors are zero either if all the system modes are local, or in the limit of high modal overlap. For low modal overlap only the statistics of the "self" terms $\psi^{(r)}_{jj}$ are important. The coupling loss factors are proportional to the damping loss factor. Furthermore, equipartition only occurs if all the modes are global. As the modal overlap increases so an increasing number of mode-pairs overlap, interact and contribute to the energy distribution within the system, and hence to the coupling loss factors. Typically, a mode overlaps and interacts with its $\pm M$ neighbours. The general situation is complicated and depends on the statistics of the natural frequencies (on which the frequency integrals Γ_{jk} (or μ_{jk}) depend) and the cross-mode participation factors $\psi_{jk}^{(r)}$ (which depend only on the mode shapes). In the high M limit the coupling loss factors asymptote to constants, with indirect coupling loss factors becoming zero. This is due to the correlation of mode shapes at shared boundaries, and the lack of correlation of mode shapes at two points which are distant from each other. The discussion for high modal overlap strictly concerned wave-bearing subsystems, for which the mode shapes have a spatially harmonic form, although the conclusions might be expected to apply in more general circumstances.

These conclusions have similarities with some previous work. The definition of local system modes in this paper can be interpreted in terms of light coupling of sets of oscillators, for which the subsystem modes and mode shapes are only slightly perturbed when the subsystems are coupled (e.g., Ref. [6], and the ad hoc assumptions of [1–4]). Here, however, modes of the whole system are used to describe the response. In Refs. [7,8] the system's behaviour was described in terms of Green functions. The system equations were seen to be in SEA form if an appropriate definition of energy was adopted—this is analogous to the treatment here where energy is used as the response variable (strictly, twice the kinetic energy) but the equations are only of SEA form if the modes in the excitation bandwidth satisfy Eq. (12). Finally in Ref. [8] the indirect coupling loss factors were seen to be negligible if the differences between the Green functions of the uncoupled and coupled systems were sufficiently small—in the terminology of this paper, this implies all modes are local.

The paper concerned broad-band frequency averages for a single system. In SEA, one should really consider (discrete frequency) ensemble averages, although one can expect the conclusions drawn here to be similar for the ensemble average behaviour, with the expectation in Eq. (12) now being found over the ensemble instead of over a broad frequency band. However, this raises the problem of how the ensemble is to be defined, both in analytical studies and in practical applications. This is by no means clear, particularly for two-dimensional systems where shape (geometric irregularity) is an issue. For example, it was seen that at low and moderate modal overlap two coupled rectangular plates behave in a quite different manner to two, coupled pentagonal plates because their modal statistics are quite different. In terms of the system modes, the ensemble definition should include natural frequency statistics (especially asymptotic modal density and spacing statistics) and mode shape statistics (both the self-terms of Eq. (12) and the cross terms $\psi_{jk}^{(r)}$) and their correlations—potentially a formidable task.

For real systems it may well be that Eq. (12) holds only for unrealistically wide frequency bands: for example if the modal density or the damping loss factor is significantly frequency dependent then that condition might never be satisfied. The response of a single realization of the system would then vary about the mean levels predicted by the ensemble (or broadband frequency) average. Furthermore, the system mode shapes might be such that the convergence to proper-SEA behaviour for large M might be slow (e.g., for nearly periodic structures). Again, the results here provide the mean about which the finite frequency band response of single realizations of a system vary. In addition, the discussion (particularly that concerning the asymptotic behaviour at high M) assumed that the contribution of out-of-band modes was small, which may well not be the case for the band average response of a single system, especially at high M.

It was also assumed that the modes had equal bandwidths. This is not strictly necessary, since averages and expectations can also be taken over modal bandwidths. Also in practice the different subsystems will usually have different damping levels, so that the results herein should be rephrased using complex system modes—it is not believed that this would substantially affect the conclusions however. Alternatively, the bandwidth for each mode could be chosen to accurately reflect the dissipation of energy, depending on the participation of each subsystem through the terms $\psi_{jj}^{(r)}$.

In conclusion it is acknowledged that practical situations to which SEA is applied are such that the system modes will rarely be known, or even able to be computed. However, the modes do exist so that the conclusions reached in this paper can be applied. Furthermore, the approach offers a general framework for the analysis, a means of estimating the SEA parameters and a basis for quantifying variability etc. in SEA predictions.

Appendix A. Modes and energy

A time harmonic point force of amplitude F acts at a point $\mathbf{x} = \mathbf{x}_1$. The amplitude of the response at point $\mathbf{x}_2$ is given in terms of the system modes by

$$W(\omega, \mathbf{x}_1, \mathbf{x}_2) = \sum_j \alpha_j(\omega)\phi_j(\mathbf{x}_1)\phi_j(\mathbf{x}_2)F, \tag{A.1}$$

where $\phi_j(\mathbf{x})$ is the mode shape of the jth system mode and where

$$\alpha_j(\omega) = \frac{1}{\omega_j^2(1 + \mathrm{i}\eta) - \omega^2} \tag{A.2}$$

is the modal receptance, ω_j being the jth natural frequency. The modes are assumed to be mass normalised so that when integrated over the whole system

$$\int \rho(\mathbf{x})\phi_j(\mathbf{x})\phi_k(\mathbf{x})\,\mathrm{d}\mathbf{x} = \delta_{jk}. \tag{A.3}$$

The amplitude of the velocity is $\mathrm{i}\omega W$, so that the time average kinetic energy density at $\mathbf{x}_2$ and the input power are

$$\begin{aligned} D_T(\omega, \mathbf{x}_1, \mathbf{x}_2) &= \tfrac{1}{2}\mathrm{Re}\{\tfrac{1}{2}\rho(\mathbf{x}_2)\omega^2 W W^*\}, \\ P_{in} &= \tfrac{1}{2}\mathrm{Re}\{\mathrm{i}\omega W(\omega, \mathbf{x}_1, \mathbf{x}_1)F^*\}, \end{aligned} \tag{A.4}$$

where $*$ denotes the complex conjugate and where $\rho(\mathbf{x})$ is the mass density.

"Rain on the roof" excitation is now assumed to act over the excited subsystem s and over a frequency band Ω. "Rain" is defined as spatially delta-correlated, random excitation with a spectral density proportional to the local mass density $\rho(\mathbf{x})$. In a frequency band $\delta\omega$ the contribution to the mean-square force is $\overline{\delta f^2} = S_f\rho(x_1)\delta\omega$, where S_f is the power spectral density (per unit mass density). While this may be a function of frequency, here it is assumed to be constant for convenience. The total kinetic energy in the responding subsystem r and the input power are then found by integrating equations (A4) over all points $\mathbf{x}_1$ in r, all points $\mathbf{x}_2$ in s and all frequencies ω in Ω. The time and frequency average kinetic energy in subsystem r

$$T^{(r)} = 2S_f \sum_j \sum_k \Gamma_{jk}\psi_{jk}^{(s)}\psi_{jk}^{(r)} \tag{A.5}$$

is thus a sum over cross-modal pairs, while the input power

$$P_{in}^{(r)} = 2S_f \sum_j 2\Delta_j \Gamma_{jj} \psi_{jj}^{(s)} \tag{A.6}$$

is a sum of modal contributions. The terms Γ_{jk} and $\psi_{jk}^{(r)}$ are defined in Eqs. (A.7) and (9), respectively. If the time average kinetic and potential energies are assumed equal, then the total energy is twice the kinetic energy. Finally, if it is assumed that all modes have equal bandwidth, then $\Delta_j = \Delta = \omega\eta$.

The integral of Γ_{jk} over the frequency range from ω_1 to ω_2 is given by

$$\Gamma_{jj} = \frac{1}{2\Omega} \mathrm{Re}\left\{ \frac{-\sqrt{-z_j}}{(z_j - z_j^*)} \arctan\left(\frac{\Omega\sqrt{-z_j}}{(\omega_1\omega_2 - z_j)} \right) \right\},$$

$$\Gamma_{jk} = \frac{1}{4\Omega} \mathrm{Re}\left\{ \frac{\sqrt{-z_j}}{(z_k - z_j)} \arctan\left(\frac{\Omega\sqrt{-z_j}}{(\omega_1\omega_2 - z_j)} \right) + \frac{\sqrt{-z_k}}{(z_j - z_k)} \arctan\left(\frac{\Omega\sqrt{-z_k}}{(\omega_1\omega_2 - z_k)} \right) \right\}, \tag{A.7}$$

where

$$z_j = \omega_j^2(1 - \mathrm{i}\eta); \quad z_k = \omega_k^2(1 + \mathrm{i}\eta) \tag{A.8}$$

and where $\Omega = \omega_2 - \omega_1$.

Appendix B. Nomenclature

$\mathbf{A}, A$	energy influence coefficient, Eq. (1); area
c_g	group velocity
$\mathbf{C}$	coupling loss factor matrix, Eq. (15)
D	plate bending stiffness
$\mathbf{E}, E$	subsystem energy; elastic modulus
$\mathrm{E}[\cdot]$	expectation operator
h	thickness
k	wavenumber
$\mathbf{L}$	damping and coupling loss factor matrix, Eqs. (2) and (3)
L	length
m	integer, difference in mode number
M	modal overlap
M_{jk}	overlap of modes j and k, Eq. (10)
n	asymptotic modal density
n_{tot}	total asymptotic modal density of the system
N	number of modes in band
N_s	number of subsystems
$\mathbf{P}_{in}$, P_{in}	input power
S	sum, Eq. (42)
$\mathbf{x}, x$	position

$\boldsymbol{\alpha}, \alpha_{rs}$	matrix, Eqs. (17) and (18)
α_j	modal receptance of mode j, Eq. (8)
Δ	modal half-power bandwidth
η	damping loss factor
η_{ij}, η_{rs}	coupling loss factor
ϕ_j	mass normalised mode shape of jth mode
Γ_{jk}	frequency integral of modes j and k, Eq. (7)
$\psi_{jk}^{(r)}$	cross mode participation factor of modes j and k in subsystem r, Eq. (9)
ρ	density
ρ_{rs}	covariance of mode participation factor in subsystems r and s
μ_{jk}	Eq. (10)
v	fractional modal density, n/n_{tot}
σ_r^2	variance of mode participation factor
ω	frequency
ω_j	natural frequency of jth mode
Ω	frequency band of excitation

Subscripts

i, j, r, s	subsystem number
j, k	mode number
tot	total for system

References

[1] C.H. Hodges, J. Woodhouse, Theories of noise and vibration transmission in complex structures, *Reports on Progress in Physics* 49 (1986) 107–170.

[2] R.H. Lyon, R.G. DeJong, *Theory and Application of Statistical Energy Analysis*, 2nd Edition, Butterworth-Heinemann, Boston, 1995.

[3] R.H. Lyon, G. Maidanik, Power flow between linearly coupled oscillators, *Journal of the Acoustical Society of America* 34 (1962) 623–639.

[4] R.H. Lyon, E. Eichler, Random vibration of connected structures, *Journal of the Acoustical Society of America* 36 (1964) 1344–1354.

[5] T.D. Scharton, R.H. Lyon, Power flow and energy sharing in random vibration, *Journal of the Acoustical Society of America* 43 (1968) 1332–1343.

[6] J. Woodhouse, An approach to the theoretical background of statistical energy analysis applied to structural vibration, *Journal of the Acoustical Society of America* 69 (1981) 1695–1709.

[7] R.S. Langley, A general derivation of the statistical energy analysis equations for coupled dynamic systems, *Journal of Sound and Vibration* 135 (1989) 499–508.

[8] R.S. Langley, A derivation of the coupling loss factors used in statistical energy analysis, *Journal of Sound and Vibration* 141 (1990) 207–219.

[9] B.R. Mace, Statistical energy analysis, energy flow models and system modes, *Journal of Sound and Vibration* 264 (2003) 391–409.

[10] B.R. Mace, The statistical energy analysis of two continuous one-dimensional subsystems, *Journal of Sound and Vibration* 166 (1993) 429–461.

[11] E.C.N. Wester, B.R. Mace, Statistical energy analysis of two edged-coupled rectangular plates: ensemble averages, *Journal of Sound and Vibration* 193 (1996) 793–822.
[12] F.F. Yap, J. Woodhouse, Investigation of damping effects on statistical energy analysis of coupled structures, *Journal of Sound and Vibration* 197 (1997) 351–371.
[13] S. Finnveden, Ensemble averaged vibration energy flows in a three-element structure, *Journal of Sound and Vibration* 187 (1995) 495–529.
[14] B.R. Mace, J. Rosenberg, The SEA of two coupled plates: an investigation into the effects of subsystem irregularity, *Journal of Sound and Vibration* 212 (1999) 395–415.
[15] B.R. Mace, Energy flow and S.E.A. at low modal overlap, *Fifth International Congress on Sound and Vibration*, Adelaide, l997.
[16] C.H. Hodges, P. Nash, J. Woodhouse, Measurement of coupling loss factors by matrix fitting: investigation of numerical procedures, *Applied Acoustics* 22 (1987) 47–69.
[17] B.R. Mace, P.J. Shorter, Energy flow models from finite element analysis, *Journal of Sound and Vibration* 233 (2000) 369–389.
[18] R. Courant, D. Hilbert, *Methods of Mathematical Physics*, Vol. 1, Interscience, New York, 1953.